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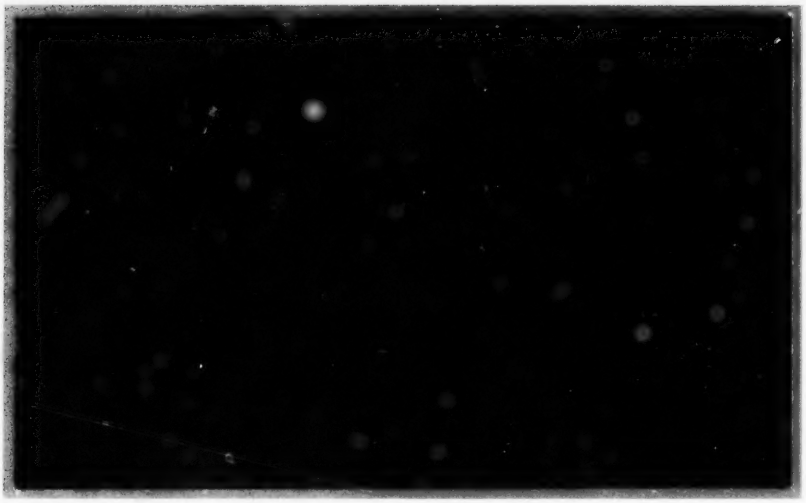

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VEHICLE ROUTING WITH UNCERTAIN DEMANDS STOCHASTIC PROGRAMMING AND ITS CORRESPONDING TSP SOLUTION

by

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ABSTRACT

In this paper we address the issue of different mathematical models for the stochastic vehicle routing problem (SVRP). This problem is inherently much more difficult than the generic deterministic vehicle routing problem (VRP) for which optimal solution procedures can solve only small problems. Presently, we cannot even begin testing optimal solution procedures for the SVRP for any problem size exceeding 3 nodes. Thus, we need to examine modeling approaches to this problem in order to exploit the structure and solution properties. We present a multistage stochastic model for the SVRP. We prove that this model has an interesting minimal graph representation in which a SVRP solution corresponds to a Hamiltonian cycle. We also present a Markov decision model for the problem, concluding with a discussion of solution prospects and directions.

Key words : Stochastic programming, vehicle routing, Markov decision process.

INTRODUCTION

The Vehicle Routing Problem - (VRP) is an important problem in the area of distribution-logistics. Following Dror et al. (1989 a) we examine here the VRP with stochastic demands - (SVRP). The SVRP is an optimization problem for which certain parameters are known in the statistical sense only. Specifically, the problem is described as follows: Given a set $N = \{0, 1,...,n\}$, where $\{0\}$ is designated as a depot and $N\backslash\{0\}$ as customers. A vehicle of a finite capacity Q resides at the depot. Each element (customer) $i \in N\backslash\{0\}$ requires that an amount $d_i > 0$ of some commodity stored at the depot be delivered to it. Let $C = (c_{ij})$ be a fixed distance matrix defined on N^2 satisfying the triangular inequality in the strong sense (i.e., $c_{ik} + c_{kj} > c_{ij}$ for all i, j, k). Assuming that d_i, $i = 1,...,n$ are known in the statistical sense, the objective is to plan an efficient (minimal distance) system of delivering from the depot the amount d_i of the commodity to each customer i. For simplicity, we assume that the exact value of d_i, $i = 1,...,n$, becomes known the instant the vehicle arrives at i and for all i $\Pr\{d_i > Q\} = 0$, $\Pr\{d_i = 0\} = 0$.

Because of its importance for physical distributions, logistics and manufacturing, the deterministic version of this problem, the VRP, has been studied extensively by numerous researchers. For a review of recent work on the VRP see Laporte and Nobert (1987), Altinkemar and Gavish (1987), and the collection of papers in Golden and Assad (1988). In spite of the great interest in the VRP, only moderate problem instances (expressed in terms of |N|) can be presently solved optimally. Note that for the VRP the demands d_i, $i = 1,...,n$ are fixed and known in advance for the purpose of planning an efficient delivery system. In addition, in the case of the generic VRP, the prevalent restriction is that each customer receives exactly one vehicle visit during which the customer's demand has to be fully supplied. (For a version of the VRP with possibility of multiple visits to a customer see Dror and Trudeau, (1989 a), (1989 b), and Dror et al. (1989 b).

Given that full demand d_i has to be delivered to each customer in all the VRP variants mentioned above, limiting the number of vehicle visits to a customer to exactly one in the case of stochastic demand

is inconsistent with generating feasible vehicle delivery routes. The uncertainty of the d_i values implies that any planned route might have to be modified at some (usually a priori unknown) intermediate point i on the route because the vehicle has less than d_i of the commodity left on its arrival to i. If that happens, then more than one visit to i will be necessary to fill its total demand. This kind of uncertainty in the execution of a preplanned vehicle delivery route is of major concern in our analysis of the SVRP. Planning efficient vehicle delivery routes with uncertain demands becomes a stochastic optimization problem of an entirely different nature than in the case of deterministic VRP.

Introducing demand uncertainty in the generic VRP setting drastically changes the difficulty/complexity of the search for optimal solutions. As we already have stated, the VRP solution procedures can solve cases with 30-40 customers. For VRPs with more customers, heuristic solution procedures have to be employed. While the VRP is a difficult problem to solve, the SVRP is inherently so much more difficult that we presently cannot even begin testing optimal solution procedures for problems larger than 2-3 customers. Thus, it seems that before embarking on the task of designing solution procedures for the SVRP, it is necessary to carefully analyze different modeling approaches to the problem in order to exploit the structure and solution properties of the problem.

In this paper we address the issue of different mathematical representations for the stochastic vehicle routing problem. We first model this optimization problem as a multistage stochastic programming problem. We prove that such a mathematical model can also be represented as a problem on a graph in which an optimal solution, in terms of the stochastic programming model, corresponds to a Hamiltonian cycle on that graph. This (our) graph is the smallest (in terms of the number of nodes and edges), for which the above Hamiltonian property holds. In the section that follows, as an alternative to the multistage stochastic programming model for the SVRP, we present a Markov decision model for the problem. We conclude with a short discussion of solution prospects for the SVRP.

Before casting this (SVRP) optimization problem in the form of Stochastic Programming we review the Stochastic Programming setting as presented in Wets (1986), and Kall (1982), while biasing the discussion to the specifics of the SVRP.

STOCHASTIC PROGRAMMING

The modeling environment in which certain problem parameters are known only in a statistical sense can be presented as follows: Given certain initial (deterministic) information $\xi_o \in R^{m_o}$, (think of it as the distance matrix (c_{ij}) and the vehicle capacity Q) a decision $x_1 \in R^{m_1}$ is taken at some cost $f_1(x_1, \xi_o)$. After executing x_1 a new information is obtained $\xi_1 \in R^{m_1}$ and a new decision $x_2 \in R^{m_2}$ is selected at a cost $f_2(x_1, x_2, \xi_{01}, \xi_1)$. This sequence of information $\Rightarrow$ decision $\Rightarrow$ information $\Rightarrow$ ... continues up to say R stages. (In our case we restrict R to be finite but in general it could be infinite.) At each stage the decisions $x_1, x_2, \ldots, x_{R+1}$ are subject to constraints that usually depend on the actual realizations $\xi_o, \xi_1, \ldots, \xi_R$.

In a quite general two-stage setting Wets (1983), models this kind of stochastic optimization problem as:

Find $x \geq 0$, $\alpha \in [0,1]$ with $P[A(\omega)x > b(\omega)] > \alpha$ such that $z(x) + \rho(\alpha)$ is minimized where $z(x) = cx + E\{\inf_{y \geq 0} q(\omega)y \mid Wy = p(\omega) - T(\omega)x\}$. "E" represents the mathematical expectation with respect to ω. The vectors b, q, p, matrices A, T are random, and vector c, matrix W are fixed. The function $\rho(\alpha)$ with $P[A(\omega)x \geq b(\omega)] \geq \alpha$ represent the chance-constrained component of the problem and $E\{\inf_{y \geq 0} q(\omega)y \mid Wy = p(\omega) - T(\omega)x\}$ represents the recourse component of the problem. Restricting the problem to fixed W is referred to as a problem with fixed recourse.

In the following discussion the chance-constraint option is not considered, since we do not intend to model the SVRP in that form but expand it as a recourse model from a two-stage to a multi-stage stochastic problem setting. We require that the decision vector(s) x be constrained to a set S, where S is a set of appropriately dimensional binary (or integer) valued vectors. Following Dantzig (1963), Wets (1966), and Birge (1985), we present a multi-stage version of a stochastic (linear integer) program with fixed recourse. We denote this problem as (MG).

(MG): minimize

$$z = c_1x_1 + E_{\xi_1}\{\min c_2x_2 + E_{\xi_2}\{\min c_3x_3 + \quad \ldots \quad + E_{\xi_T}\{\min c_{T+1}x_{T+1}\}\ldots\}\}$$

subject to:

$$\begin{aligned}
A_1x_1 &= \xi_0 \\
-B_1x_1 + A_2x_2 &= \xi_1 \\
-B_2x_2 + A_3x_3 &= \xi_2 \\
&\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\
-B_Tx_T + A_{T+1}x_{T+1} &= \xi_T
\end{aligned}$$

$x_t \geq 0$ and integer, $t = 1,\ldots,T+1$

c_t is a known vector in R^{n_t}, $t = 1,\ldots,T+1$

ξ_o is a known vector in R^{m_o}, ξ_t is a random m_t vector defined on the probability space (Ω_t, Σ, P), $t = 1,\ldots,T$. We assume that Ω_t is bounded with probability 1 for all t. A_t, B_t are correspondingly dimensioned real valued matrices. The vector x_1 represents decisions made in the first period. Future decision vectors x_t depend on previous realizations of $\xi_1,\ldots,\xi_{t-1}$, decisions $x_1,\ldots,x_{t-1}$ and the distribution parameters of $\xi_t,\ldots,\xi_T$ as well as the problem's (static) constraints. The way Dantzig (1963), states it; $x_1 \geq 0$ is chosen to satisfy $A_1x_1 = \xi_o$ (it is assumed that the feasibility/existence of x_t, $t = 1,\ldots,T+1$ is never in question), $x_2 \geq 0$ is chosen to satisfy $-B_1x_1 + A_2x_2 = \xi_1$ after ξ_1 is drawn randomly as a point from Ω_1, and this process continues till $x_{T+1} \geq 0$ is chosen. The problem is to determine the first period decision x_1 that minimizes the current cost c_1x_1 plus the expected values of all future costs in periods 2,..,T+1.

PROPERTIES OF THE SVRP SOLUTION

Before modeling the SVRP as the kind of stochastic programming model presented above, we examine some properties of an optimal SVRP solution based on assumptions about the customer demand

distributions. For this analysis, there are two important assumptions about customer demands: (i) $Pr\{d_i > Q\} = 0$, i=1,...,n, (i.e., no individual demand is greater than the vehicle capacity), (ii) if a vehicle visits a customer it has to deliver to that customer whatever fraction of the customer's demand the vehicle carries on arrival and if the vehicle is empty it has to return directly to the depot. (The interpretation is that visits to a customer just for the sake of collecting the information on the customer's demand is not an acceptable service policy; this assumption precludes the option of preemption during a refill of customer's demand; i.e., a refill has to be as complete as possible). Under these restrictions, we prove that

Proposition 1: Given SVRP with the assumption (ii) and a distance matrix with triangular inequality satisfied in the strong sense, then any arc (i, j); $j \neq 0$ is traversed at most once in an optimal SVRP solution.

Proof: Assume arc (i, j), $j \neq 0$ is traversed in an optimal SVRP solution. Given service restriction (ii) the demand d_i at i is fully satisfied before the vehicle can continue to j, $j \neq 0$ (otherwise the vehicle returns to the depot, thus $j = 0$). If the vehicle needs to revisit j and $i \neq 0$, (d_j was not fully satisfied on the previous visit from i), then it has to arrive at j from some node k. But since $c_{ki} + c_{ij} > c_{kj}$ for any k = 0,1,...,n; $k \neq i$ and no additional delivery is needed at i, the vehicle will arrive at j from $k \neq i$.

This optimal solution property allows the decision variables x_{ij} ($j \neq 0$) to be restricted to binary values taking the value one if a vehicle travels directly to j after i and otherwise taking the value zero.

Proposition 2: The number of trips to the depot in an optimal SVRP solution is less than or equal to n.

Proof: The maximal number of possible trips back to the depot includes: (a) the route failures (return trips because the vehicle did not have enough of the commodity to fill a customer's demand), and (b) decisions to return the vehicle to the depot after replenishing a customer's demand (possibly in anticipation of route failure at a less favorable point down the route). Since Pr { $d_i > Q$} = 0 for all i = 1,...,n then on each vehicle route at least one customer is fully replenished. (Note that no preemption is allowed.) Thus,

$$\sum_{i=1}^{n} x_{i0} \leq n.$$

In addition, we prove the following:

Proposition 3: In an optimal SVRP solution no customer can be visited more than n-1 times (even if the best decision after each visit is to return to the depot and refill the vehicle).

Proof: It is clear from Proposition 2 that the number of visits to any $i \neq 0$ in an optimal SVRP solution is less than or equal to n. For i = 0 this number can reach n (i.e., the maximal number of times a vehicle will return to the depot is n), but for $i \neq 0$ we (i.e., a customer) prove that the number of visits can be at most n-1.

There are n customers and the first one is always fully replenished in one visit. There are at most n-1 additional vehicle departures from the depot which in principle could mean n visits to one single customer. At this point the triangular inequality excludes one visit in an optimal solution. The exact customer's demand becomes known on the first visit to a customer and if it is determined that on the n-1 visit the vehicle would not be able to complete the replenishment of a given customer, but would need one more trip just for that customer, then the vehicle can return directly to the depot to refill, skipping the n-1st visit. Thus, altogether there could be at most n-1 visits to any customer.

GRAPH REPRESENTATION FOR THE SVRP

Following the observations regarding the properties of any optimal SVRP solution, as described in Propositions 1, 2 and 3, we present an expansion of the original (complete) graph on which we seek to construct an optimal SVRP solution. The 'new' expanded graph is a minimal graph with the property that any optimal stochastic vehicle routing solution of the original problem corresponds to a Hamiltonian cycle (a cycle which passes through each node exactly once), on the 'new' graph. Moreover, any Hamiltonian cycle on the 'new' graph, given a realization of demands which make such a solution feasible (in terms of vehicle capacity), describes an SVRP solution which satisfies Propositions 1, 2 and 3 in terms of its properties. The 'new' graph is constructed as follows:

Since in the new graph there are n nodes corresponding to the depot and n-1 nodes for each customer, the convention should be that if for a given demand realization only k, $1 \leq k \leq n$, vehicle tours are required in the optimal tour, then n-k+1 nodes corresponding to the depot are directly connected in such tour. If a node requires k, $1 \leq k \leq n-1$, visits than n-k duplicates in such node are directly interconnected in an optimal tour.

Duplicate each customer node n-2 additional times (i.e., for each customer there are n-1 nodes), and the depot node n-1 additional times. Thus, the new graph contains $n(n-1) + n = n^2$ nodes all together. All n-1 nodes which represent one customer are interconnected by arcs of zero distance and connected to the other nodes by arcs with the same distance as in the original graph. The nodes in the new graph are renumbered as follows: The n nodes from 0 to n-1 correspond to the depot in the original graph, nodes from n to 2(n-1) correspond to node 1 in the original graph, nodes 2(n-1) + 1 to 3(n-3) correspond to node 2 in the original graph, nodes k(n-1) + 1 to (k + 1) (n-1) correspond to node k in the original graph for k = 1, ..., n. The n depot nodes are also interconnected with arcs of zero distance and connected to the other nodes by arcs from the original distance matrix. On this new graph we will define a Hamiltonian tour (a traveling salesman tour) which corresponds to an SVRP solution (see illustration in Figure 1).

Figure 1 about here

In order to model the SVRP problem as an n-stage stochastic programming problem we present the following notation:

Denote by π a permutation of numbers $\{0, 1,...,n^2-1\}$ (i.e., $\pi:\{0, 1,...,n^2-1\} \rightarrow \{0, 1,...,n^2-1\}$ is a one-to-one mapping from the set onto itself with the following property:

$\pi(0) = 0$, and $\pi(0) < \pi(1) < ... < \pi(n-1)$ (the depot node)

$\pi[k(n-1) + 1] < \pi[k(n-1) + 2] < ... < \pi[(k + 1)(n-1)]$, for $k = 1, ..., n$ (the customer nodes).

This implies that we restrict the permutation π in a way that preserves the relative order of the set of nodes in the new graph representing a node in the original graph (depot or customer). If customer's demand has been fully supplied, say on the ℓ^{th} visit ($1 \leq \ell \leq n-1$), then we will visit immediately (at no cost) all the rest of the nodes corresponding to that customer in the order consistent with the property of π.

Let $\pi^1 = \min \{\pi[k(n-1) + 1];\ k = 1, ..., n\}$ and similarly denote by $\pi^i = \min \{\{\pi[k(n-1) + 1]\} \setminus \{\pi^1, \pi^2, ..., \pi^{i-1}\};\ k = 1, ..., n,\ i = 2, ..., n\}$.

Set $\pi^{n+1} = n-1$.

The interpretation of π^i is that the permutation π selects the customer π^1 to be visited first, π^2 second, etc. Note that after the first visit to a customer, for all the following visits to that customer, the demand is known with certainty.

Next, we model the SVRP as a stochastic multi-stage program with fixed recourse. The number of stages in such an SVRP model is determined by the number of times new demand information is revealed; thus, it has exactly n stages (the number of customers).

STOCHASTIC PROGRAMMING MODEL

Now we can state the SVRP as a multi-stage linear stochastic programming model with binary variables of exactly n stages denoted by (P) as follows:

minimize

$$z = c_{0\pi^1} x_{0\pi^1} + E_{d_{\pi^1}} \{\min(c_{\pi^1\pi(i)} x_{\pi^1\pi(i)} + \sum_{\pi(i)<\pi^1}^{\pi^2-1} c_{\pi(i)\pi(i)+1} x_{\pi(i)\pi(i)+1}) + E_{d_{\pi^2}} \{\min(c_{\pi^2\pi(i)} x_{\pi^2\pi(i)} +$$

$$\sum_{\pi(i)<\pi^2}^{\pi^3-1} c_{\pi(i)\pi(i)+1} x_{\pi(i)\pi(i)+1}) + \ldots + E_{d_{\pi^n}} \{\min(c_{\pi^n\pi(i)} x_{\pi^n\pi(i)} + \sum_{\pi(i)<\pi^n}^{\pi^{n+1}-1} c_{\pi(i)\pi(i)+1} x_{\pi(i)\pi(i)+1})\}\ldots\}\} \quad (1)$$

subject to:

$$\sum_{i=0}^{n^2-1} x_{\pi(i)\pi(j)} = 1 ; \qquad j=1,\ldots,n^2-1 \ (i \neq j) \quad (2)$$

$$\sum_{j=0}^{n^2-1} x_{\pi(i)\pi(j)} = 1 ; \qquad i=1,\ldots,n^2-1 \ (i \neq j) \quad (3)$$

$$[x_{\pi(i)\pi(j)}] \in S \quad (4)$$

$$\sum_{\substack{j=1 \\ j\neq i}}^{n^2-1} \sum_{\pi(i)=\pi^k}^{\pi[(k+1)(n-1)]} f_{\pi(j)\pi(i)} - \sum_{\substack{j=1 \\ j\neq i}}^{n^2-1} \sum_{\pi(i)=\pi^k}^{\pi[(k+1)(n-1)]} f_{\pi(i)\pi(j)} = d_{\pi^k} ; \qquad k=1,\ldots,n \quad (5)$$

$$f_{\pi(i)\pi(j)} \leq Q x_{\pi(i)\pi(j)} ; \qquad i,j=0,\ldots,n^2-1 \quad (6)$$

$$x_{\pi(i),\,\pi(j)} = 0 \text{ or } 1; \qquad i,j=0,\ldots,n^2-1 \quad (7)$$

$$f_{\pi(i),\,\pi(j)} \geq 0 ; \qquad i,j=0,\ldots,n^2-1 \quad (8)$$

Expression (1) is our objective function which expresses the cost of the first stage decision plus the expected cost of the decisions in the other n-1 stages. Each stage starts with new demand information. The optimization is over all permutations of $\{0,\ldots,n^2-1\}$ with the properties stated above. Each node in the 'expanded' graph is visited exactly once. Constraints (2) and (3) imply that if at some visit we fully replenish a given customer, then we will immediately traverse all the remaining (not yet

visited) nodes which belong to that customer at zero cost. We can, of course, also visit the depot and the customers whose demand is already known on our way to a new demand point - new stage of this model. Constraints (4) are the symbolic version of the subtour elimination constraints for the traveling salesman problem. Constraints (5) state that the demand of the first customer in the kth stage in the permutation π is fully satisfied through the flow variables $f_{\pi(i)\pi(j)}$. Note that at most n-1 visits to any customer are allowed (the first summation in each of the two parts of the flow expression). Constraints (6) simply state that the flow is allowed only through the arcs selected in the solution. We assume that each time the vehicle leaves the depot it carries a full load. Constraints (7) are the usual binary requirements on the decision variables and (8) are the nonnegativity requirement on the flow variables.

It is not easy to see how the model (P) can be transformed into a 'lower diagonal' structure that the (MG) model has, but nevertheless it expresses the same kind of a multi-stage stochastic optimization setting.

Setting aside the objective function in problem (P), the constraints set ((2) - (8)) follows very closely the familiar deterministic vehicle routing problem model in Laporte and Nobert, (1987). The difference between this set of constraints and the traditional one is in the size of the corresponding graph which is expanded in formulation (P) from an n+1 node complete graph with corresponding distance matrix satisfying a triangular inequality, into an n^2 complete graph with a special distance structure between its nodes. The constraints (2) - (4) and (7) indicate that the optimal solution has to correspond to a Hamiltonian cycle on the expanded graph satisfying the additional flow constraints expressed by (5), (6) and (8). However, the main difficulty when contemplating solution approaches for this stochastic programming model lie in the functional form of its objective. At this junction further research is necessary in order to develop computational solution strategies for this formulation.

SVRP as a Markov Decision Process Model

A different approach to solving the SVRP is to view the problem not in the stochastic programming framework but as a Markov decision process type of a problem.

In this section we essentially restate a Markov Decision Process model for the SVRP presented in Dror et al. (1989 a) with some modifications to represent the SVRP assumptions of this paper. We start with the assumptions.

Consider an SVRP with a single vehicle of fixed capacity Q located at the depot and assume that no customer demand exceeds Q. Assume that the exact demand becomes known immediately upon arrival of the vehicle at the customer location (i.e., before replenishment begins). Once a vehicle arrives at customer location it has to fully supply the demand of that customer or as much of the commodity needed as the vehicle has available. Given the vehicle arrived at a customer location only one decision-action has to be taken: replenish the customer and then move to another location. Since a nonpreemptive service policy is assumed, only in case the vehicle is empty or all customers have been fully replenished does the vehicle return to the depot. In the initial state there is a full vehicle at the depot with all n customers to be serviced. The final state of the system is when no more customers need to be replenished; in this case the vehicle has to return to the depot. The system is observed each time the vehicle arrives for the first time at one of the locations of {0, 1,...,n}. Let $\tau_0 = 0, \tau_1, \tau_2;...,\tau_n$ ($\tau_i \leq \tau_{i+1}$; i = 0,1,...,n) be the times (total distance traveled by the vehicle) at which these events occur. Note that τ_i corresponds to the time of the ith arrival which is not necessarily an arrival time at location i. These are called transition times and correspond to the times at which decisions are taken. The state of the system at a transition time τ_k is described by a vector $s = (r, l, x_1,...,x_n)$, where $r \in \{0, 1,...,n\}$ denotes the position of the vehicle and $l \in [0, Q]$ describes the commodity level in the vehicle (if r = 0 then $l = Q$, i.e., the vehicle is full). If customer i has been visited, then the exact demand is known and x_i represents the remaining demand ($0 \leq x_i < Q$). If customer i has not been visited yet, then the demand is unknown and is denoted by $x_i = -1$. The state space is a subset S of $\{0, 1,...,n\} \times [0, Q] \times (\{-1\} \cup [0, Q))^n$ which satisfies the above conditions.

At each transition time, a decision d is selected from the decision space $D = 2^{\{1,...,n\}} \times \{0, 1,..,n\}$, where d = (P, i) and $P \in 2^{\{1,...,n\}}$ means that the vehicle goes to a customer whose demand is yet undetermined i from its present location, say j, and on its route to i replenishes, on a shortest path,

a subset of customers P whose demand is already known. (Note that P could be ϕ.) The decision $d = (\cdot,i)$, $i \neq 0$ is admissible only if $l \in (0, Q]$, and $d = (\cdot,0)$ whenever $l = 0$ or $x_i = 0$ for $i = 1,...,n$. For each $s \in S$, let $D(s)$ D denote the set of admissible decisions when the system is in state s, and $\Gamma = \{(s,d) | d \in D(s)\}$ the set of admissible state-decision pairs.

At transition time τ_k, the system is in some state $s_k \in S$ and a decision $d_k \in D(s_k)$ is taken. The time τ_{k+1} of the next transition is deterministic (specified by the distance matrix (c_{ij})). The next state s_{k+1} is generated according to the probability distribution which governs the demand variables. Suppose $s_k = (\tau_k, l, x_1,...,x_n) \in S$ is the observed state at the current transition time τ_k, and $d \in D(s)$ is the decision taken. Then the time (distance) until the next event is simply the difference $\tau_{k+1} - \tau_k = c_{ji}$ where $r_{\tau_k} = j$ is the location of the vehicle at transition time τ_k, $r_{\tau_{k+1}} = i$ and $d = (\phi,i)$. In case $d = (p \neq \phi, i)$, then the time-distance until the next event $\tau_{k+1} - \tau_k$ = shortest path distance through set P ending at location i. We implicitly assume that service time is zero. Let $p(\cdot | s,d)$ be the transition law, that is for every Borel subset $\overline{S}$ of S, $p(\overline{S}|s,d)$ is the probability that the next state belongs to $\overline{S}$, given s and d.

A control policy is a function μ which associates with each state $s \in S$ a decision $d = \mu(s) \in D(s)$. The aim of the decision maker is to find a policy which minimizes the expected distance of a decision sequence, starting at $s_o = (0, Q, -1,...,-1)$ and terminating at $s^* = (0, Q, 0,...,0)$. More precisely, one seeks a policy μ^* which minimizes

$$E_\mu [\sum_{k=0}^{n} \bar{c}_{r_{\tau_k}, r_{\tau_{k+1}}}]$$

wher

e n is the number of transitions and $\overline{c}_{ji}$ is the usual distance c_{ji} in case at the transition time τ_k ($r_{\tau_k} = j$) the decision $d = (\phi,i)$ or is the length of the shortest path from j to i through the set P if the decision is $d = (p \neq \phi,i)$.

DISCUSSION

The SVRP models presented in this paper - the stochastic programming model (P) and the Markov decision process model - are both general enough to be of considerable interest and at the same time restricted enough for the purpose of clarity of the solution properties and decision options. Since both models examine the same problem under the same demand distribution assumptions, information availability, and policy options, they are equivalent in terms of optimal solutions. Wets (1966), presents a multi-stage stochastic problem which under the assumptions that (i) the solution vectors x belong to a convex set and (ii) the cost functions ($c^i(x^i)$ in Wets terms) are convex functions for i = 1,...,n. Wets goes on to prove that this multi-stage problem is equivalent to that of finding a vector x which minimizes a convex function under linear constraints. (See also Dantzig, 1963, Chap. 25). Obviously, obtaining an explicit, form of such a convex function is not practical, but yet this result is significant. Since for the SVRP neither (i) nor (ii) are satisfied, the question is whether Wets' conclusion still holds. In fact, it does, and we simply restate the argument from Dantzig 1963, Chap. 25.4: "The fact that a cost function for the (m-1)st stage can be obtained from the mth stage is simply a consequence of the fact that optimal behavior for the mth stage is well defined; that is, given any stage, $(X_1, X_2,...,X_{m-1})$ at the beginning of this stage, the best possible actions can be determined and the minimum expected cost evaluated." Clearly, this reduction argument follows through to the first stage. In the SVRP case there is no convexity of either the solution set nor of the cost functions but the reduction argument basically of the dynamic programming flavor is valid nevertheless.

An optimal solution to the multi-stage stochastic programming model (P) does not seem presently attainable. What might be of potential comfort is the fact that solving each stage of this model (P) is equivalent to solving a stage of the Markov decision process - (MDP) model: this, indicates that a solution procedure for such an SVRP is more approachable from the MDP prospective. As observed in L'Ecuyer (1988), even with this MDP approach, general solutions are presently unlikely and an optimistic statement would be that we could solve three node problems with a lot of computational effort. Presently, the research of SVRP type problems is limited to quite restricted versions in which the prevailing assumption is that no reoptimization of a route is allowed when demand becomes known,

even in the event of route failure. (That is, the route is followed as initially designed regardless of the demand realizations of the customers.) In the case of an anticipated route failure or in the actual occurrence of route failure, the usual recourse assumed in the literature is a back and forth trip to the depot from the node at which failure occurred Laporte et al. (1989), Laporte and Louveaux (1987), Dror and Trudeau (1986), Dror et al. (1989 a). Future research directions should attempt to approach the problem more completely, with some form of route reoptimization upon new demand information. Since such a task is quite hard, the other option is to investigate other kinds of recourse actions than those assumed in past research. In addition, clear classification of routing problems with uncertainty in terms of decision space-reoptimization, information availability, service policies, etc., should be of considerable value.

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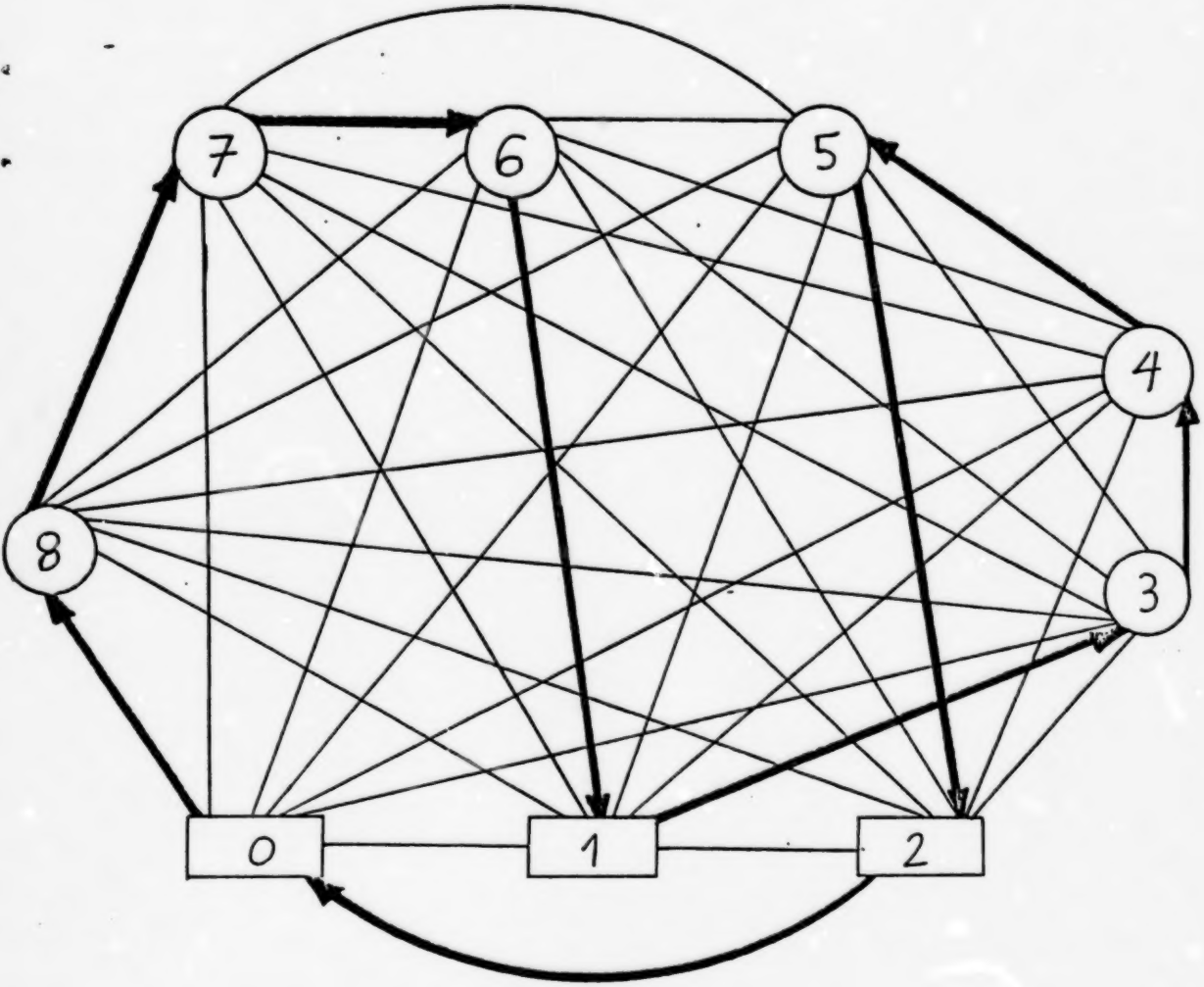

Figure 1: The expanded graph for a depot (nodes (0,1,2)) and three customers (nodes (3,4), (5,6), and (7,8)) together with a traveling salesman solution corresponding to one route failure at customer (5,6).